Load in CDMA Cellular Systems and its Relation to the Perron Root

Christian Fischer, *Member, IEEE*

***Abstract*— It is well known that the feasibility of the power allocation problem in CDMA systems is directly linked to the Perron root of a channel gain matrix [1]. In this paper, we show how the concept of load, as used in the CDMA network planning community, is linked to the Perron root and propose to use the Perron root as a load estimate. Specifically, we propose a practical iterative method to estimate the Perron root and revise various other approaches. All methods are analysed with respect to their suitability for analytical use in early-stage network planning. In addition, we extend the use of these methods from primitive matrices to cover the general case of non-negative channel gain matrices and illustrate the use of the irreducible normal form to provide tighter bounds than direct application under certain circumstances. In contrast to the use of traditional load equations, the proposed methods do not require the solution of the power allocation problem. Numerical examples show that particularly the proposed iterative solution provides very good results at low complexity.**

***Index Terms*— Code division multiaccess (CDMA), UMTS, system load, Perron root, largest eigenvalue estimation, spectral radius, non-negative irreducible primitive matrices, normal form, Gershgorin, Brauer, Brualdi.**

I. Introduction

In CDMA-based cellular systems like UMTS, link budgets are widely used for early-stage network planning to estimate both coverage and capacity [2]. An important term in these link budgets is the *system load*, or simply *load*. For universal reuse systems like CDMA, the load is introduced in the link budget to account for the multi-user interference arising from the lack of orthogonality between users in time and frequency [2] and inter-cell interference. Hence, the load is important to assess the cellular system's traffic handling abilities, but also effects such as cell breathing.

The problem with using the classical load formulae introduced in Section II is that they require the power allocation. Hence, the network planner first needs to find the Perron root of the channel gain matrix to establish that a solution to his scenario exists and then solve the power allocation problem to finally be able to compute the resulting load. However, the power allocation problem is often impracticable for first capacity approximations and early-stage network planning. Therefore, the link budgets are generally constructed using a worst-case load value for which the system is known to work in general, without a direct link to actual user numbers, services required etc. In this paper, we suggest that the Perron root itself can be used as a load estimate and propose different low-complexity estimation methods. This approach allows to directly associate the load to a given scenario while bypassing the power allocation problem, resulting in reduced complexity.

In earlier related work, e.g. [3], [4] and in particular the important paper by Hanly [1], the significance of the Perron root as a congestion measure and its application to the power allocation problem and power warfare was established and analysed in-depth. However, the link between the system load and the Perron root has not before been established. In addition, the important question of how to estimate the Perron root, and hence the existence of a feasible power allocation, has largely gone unaddressed, in contrast to the well-studied power allocation problem.

We begin in Section II by extending the classic load equations to the general multi-cell system and show the precise relationship to the power allocation problem and the Perron root in Sections III and IV. In Section V, we extend these results beyond Perron-Frobenius theory to accommodate the more general case of finding the spectral radius of general non-negative matrices. We then revise, in Section VI, various available results from matrix theory to estimate the Perron root that lend themselves to analytical use for early network planning. Furthermore, we show how the use of the irreducible normal form of the gain matrix can improve the tightness of the spectral radius bound of some standard techniques under certain circumstances. We conclude with numerical examples to compare the proposed methods in Section VII.

II. Load Equations for Network Planning

In CDMA, an important factor to assess coverage and capacity restrictions through link budgets is the *system load* [2], [5]. A load is generically defined as the ratio between used resources and totally available resources. In a system like GSM, the definition of load is straightforward and can be defined as the ratio of time slots used to time slots available. In UMTS, the question of load is not as evident because the system is not capacity hard-limited like GSM and entire theses have been written on the topic [6]. In practice, the definitions in [2] and [5] are widely used. Following the approach of [2], we derive the load equations for the general multi-cell CDMA system in the following Section. It is shown that the load thus derived quantifies the fraction of the total transmit power that is required to overcome the effects of multiuser interference. These load equations are then shown to specialise to the traditional load equations in the single cell case [2], [5].

A. Downlink Load

The performance of any one user k in the downlink (or uplink) depends on its E_b/I_0, the energy per bit over the interference plus noise density. The energy per bit is given

Manuscript received December 3, 2005; revised July 15, 2006 and October 11, 2006; accepted October 23, 2006. The associate editor coordinating the review of this paper and approving it for publication was J. Andrews.

C. Fischer is with the Department of Electrical and Computer Engineering, University of Illinois, Chicago, IL, 60607 USA, and Fischer AG Telecom Engineering, 5210 Windisch, Switzerland (email: fischerc@uic.edu).

Digital Object Identifier 10.1109/TWC.2007.05982.

by $E_b = E_c \frac{W}{R}$ where E_c is the energy per chip, W is the bandwidth in Hz and R is the data rate in bits/second. The ratio $\frac{W}{R} \triangleq g$ is called the spreading gain. We define the Signal-to-Interference-plus-Noise ratio, SINR, by $SINR = \frac{1}{g} E_b / I_0$.

Consider a downlink scenario with M base stations and K users in total, each user k is connected to a single serving base station $m_k \in \{1, \ldots, M\}$, i.e., soft handover is not considered. For a particular user k, the SINR at the user equipment/mobile is given by

$$SINR_k = \frac{P_k L_{k,m_k}}{\sum_{\substack{n=1\\n\neq k}}^{K} P_n L_{k,m_n} (1 - \alpha_k \delta_{m_k}(m_n)) + \sigma_v^2} \tag{1}$$

where P_k is the transmitted power of user k at its serving base station m_k, L_{k,m_k} is the path loss from base station m_k to user k, σ_v^2 is the AWGN power, $\alpha_k \in [0, 1]$ is the orthogonality factor in the downlink channel to user k, where $\alpha = 1$ indicates perfect orthogonality, i.e. an AWGN channel. α_k is a function of the multipath conditions and appropriate average values for different operating environments are known to the network planners from experience [2]. $\delta_{m_k}(m_n)$ denotes the indicator function such that $\delta_{m_k}(m_n) = 1$ if and only if $m_n = m_k$ and zero otherwise. The indicator function is used to take the orthogonality factor α_k into account for users served by the same base station m_k, since these users are synchronous and traverse the same channel. Interfering users that are connected to a different base station are considered non-orthogonal.

Solving (1) for P_k and then summing over all users k, we can write the total transmit power $\mathcal{P} = \sum_{k=1}^{K} P_k$ as

$$\mathcal{P} = \frac{\overbrace{\sigma_v^2 \sum_{k=1}^{K} \frac{SINR_k}{L_{k,m_k}}}^{P_{noise}}}{1 - \underbrace{\frac{1}{\mathcal{P}} \left[\sum_{k=1}^{K} \frac{SINR_k}{L_{k,m_k}} \left(\sum_{\substack{n=1\\n\neq k}}^{K} P_n L_{k,m_n} \left(1 - \alpha_k \delta_{m_k}(m_n)\right) \right) \right]}_{\eta_{DL}}} \tag{2}$$

where η_{DL} is the downlink load expression, i.e. the ratio of the transmitted power due to multi-user interference over the total transmit power. Therefore, $0 \leq \eta_{DL} < 1$. Hence, the total transmit power can be written as an increase over the power required for single-user AWGN channels as $\mathcal{P} = P_{noise} \frac{1}{1-\eta_{DL}}$ and the extra transmit power required due to multiuser interference in the link budget is $-10 \log_{10}(1 - \eta_{DL})$.

By writing (2) in terms of each base station, we can obtain a more familiar form of the downlink load. Consider the summation terms of the set of all the users that are served by base station m_k as $\Pi(m_k)$, we see that

$$\begin{aligned}\Pi(m_k) &= \sum_{\substack{n=1; n\neq k\\ \{n: m_n = m_k\}}}^{K} P_n L_{k,m_n} (1 - \alpha_k \delta_{m_k}(m_n)) \\ &= L_{k,m_k} \underbrace{(1 - \alpha_k)\beta_{m_k}}_{1-\alpha'_k} \mathcal{P}\end{aligned} \tag{3}$$

where $\beta_{m_k} = \sum_{\substack{n=1; n\neq k\\ \{n: m_n = m_k\}}}^{K} P_n / \mathcal{P}$ is simply the total transmit power at base station m_k as a proportion of the total transmit power $\mathcal{P}$. If furthermore, we write the total transmit power from each base station $m_n \neq m_k$ as the sum of its user powers, we find:

$$\begin{aligned}\forall m_j \neq m_k : \Pi(m_j) &= \sum_{\substack{n=1; n\neq k\\ \{n: m_n = m_j\}}}^{K} P_n L_{k,m_n} (1 - \alpha_k \delta_{m_k}(m_n)) \\ &= L_{k,m_j} \sum_{\substack{n=1\\ \{n: m_n = m_j\}}}^{K} P_n = L_{k,m_j} \beta_{m_j} \mathcal{P}\end{aligned} \tag{4}$$

By substituting (3) and (4) back into (2), we obtain the more familiar form of η_{DL} as it can be found in e.g. [2], [5].

$$\eta_{DL} = \sum_{k=1}^{K} \frac{SINR_k}{L_{k,m_k}} \left(L_{k,m_k} (1 - \alpha'_k) + \sum_{\substack{j=1\\ j\neq m_k}}^{M} L_{k,m_j} \beta_{m_j} \right)$$

B. Uplink load

The derivation of the uplink load follows the same approach as shown above for the downlink load. The SINR for user k in the uplink is given by

$$SINR_k = \frac{P_k L_{k,m_k}}{\sum_{n=1}^{K} P_n L_{n,m_k} - P_k L_{k,m_k} + \sigma_v^2} \tag{5}$$

where P_k is the transmitted power of user k at its mobile terminal. Rearranging (5) for the *received* power of user k and summing over all users k, we can write the total received power $\mathcal{P}' = \sum_{k=1}^{K} P_k L_{k,m_k}$ as

$$\mathcal{P}' = \frac{\sigma_v^2 \sum_{k=1}^{K} \frac{SINR_k}{1+SINR_k}}{1 - \underbrace{\frac{1}{\mathcal{P}'} \left[\sum_{k=1}^{K} \frac{SINR_k}{1+SINR_k} \left(\sum_{n=1}^{K} P_n L_{n,m_k} \right) \right]}_{\eta_{UL}}} \tag{6}$$

As for the downlink, equation (6) applies to the general multi-cell case. We see that the uplink load is the ratio of the received signal power due to multiuser interference over the total received signal power and therefore $0 \leq \eta_{UL} < 1$. In the single cell case ($m_k = m_n \ \forall m_n, m_k$), it can be shown that we obtain the familiar uplink load expression corresponding to equations (4.15) and (4.16) in [2]:

$$\eta_{UL} = \sum_{k=1}^{K} \frac{SINR_k}{1 + SINR_k}$$

III. Feasibility of the Power Allocation Problem

In deriving the load equations (2) and (6), we implicitly assumed that we know the user power allocation necessary to satisfy the SINR requirements of all the users. In practise however, the solution to the power allocation problem is not known for planning purposes. The power allocation problem has been considered in greater detail in e.g [1], [4], [7] and we only summarise the important aspects to our problem here. To service all users, we require $SINR_k \geq S_k \ \forall k$ where $S_k > 0$

is some target SINR. The resulting set of inequalities from (1) and (5) is equivalent to

$$(\mathbf{I}_K - \mathbf{H})\mathbf{p} \geq \sigma_v^2 \mathbf{s} \tag{7}$$

elementwise. We have defined $\mathbf{p} = [P_1, \ldots, P_K]^T$, $\mathbf{s} = [S_1/L_{1,m_1}, \ldots, S_k/L_{k,m_k}]^T = [s_1, \ldots, s_k]^T$, $\mathbf{I}_K$ is the $K \times K$ identity matrix and $\mathbf{H}$ is a $K \times K$ matrix with elements

$$h_{kn} = \begin{cases} 0 & \text{if } k = n \\ \frac{S_k L_{k,m_n}\left(1-\alpha_k \delta_{m_k}(m_n)\right)}{L_{k,m_k}} & \text{if } k \neq n \end{cases} \tag{8}$$

for the downlink. For the uplink problem, $\mathbf{H}$ is a $K \times K$ matrix with elements

$$h_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{S_i L_{j,m_i}}{L_{i,m_i}} & \text{if } i \neq j \end{cases}$$

A *feasible* solution to (7) is given when a solution can be found for $\mathbf{p} > \mathbf{0}$, where $\mathbf{p}$ is elementwise greater than $\mathbf{0}$. The elements $h_{kn},\ k \neq n$ are strictly positive since both the SINR constraint and the pathloss are positive. Therefore, $\mathbf{H}$ is a *nonnegative* matrix. Further, since $\mathbf{I} + \mathbf{H} > \mathbf{0}$ then $(\mathbf{I} + \mathbf{H})^{K-1} > \mathbf{0}$ and hence $\mathbf{H}$ is *irreducible* nonnegative [8].

For $\mathbf{H}$ nonnegative irreducible, the *Perron-Frobenius theorem* states that there exists an eigenvector $\mathbf{v}_1 > \mathbf{0}$ corresponding to the uniquely largest, nonnegative and real eigenvalue, the Perron root, so that $\mathbf{H}\mathbf{v}_1 = \lambda_{max}\mathbf{v}_1$, $\lambda_{max} = \rho(\mathbf{H})$ [8], where $\rho(\mathbf{H}) = \max_i |\lambda_i|$ is the spectral radius of $\mathbf{H}$. In particular, the power allocation problem in (7) has a solution for $\mathbf{p} > \mathbf{0}$ if and only if $\lambda_{max} < 1$, in which case the optimal solution is given by $\mathbf{p}_0 = \sigma_v^2(\mathbf{I} - \mathbf{H})^{-1}\mathbf{s}$. The constraint on $\lambda_{max} < 1$ is necessary and sufficient only in the absence of power constraints, as is the case in this paper. For practical systems, appropriate elementwise power constraints (uplink) and/or a sum constraint (downlink) on the elements of $\mathbf{p}$ would need to be satisfied additionally.

IV. Link Between the Perron Root and the Load

In Section II we have found that the load is $0 \leq \eta < 1$. From practical experience, network planners know that a practical system load has to be limited to some $\eta' < 1$. Given that a feasible solution to the power control problem in (7) requires the Perron root $\lambda_{max} < 1$, one may ask whether the load is related to λ_{max} and if so, whether the system load could be estimated based on results from matrix theory. In this section, we take a closer look at the role of the Perron eigenvector in the power allocation problem and show why the associated Perron root is a reasonable estimator for the load.

Proposition 1: The optimal solution to the power allocation problem $\mathbf{p}_0$ always contains a non-zero component in the direction of the Perron eigenvector $\mathbf{v}_1$. As $\lambda_{max} \to 1$, $\mathbf{p}_0 \to \alpha\mathbf{v}_1$ for some positive scalar α. Since $\eta = \lambda_{max}$ for a power allocation of $\alpha\mathbf{v}_1$, the load η can be estimated from λ_{max} with increasing accuracy as $\lambda_{max} \to 1$.

Proof: The basic process is to decompose the optimal power allocation $\mathbf{p}_0$ into a component in the direction of the Perron eigenvector $\mathbf{v}_1$ and an orthogonal complement. It is then shown that the angle between the Perron eigenvector and the optimal power allocation tends to zero as the Perron root tends to unity.

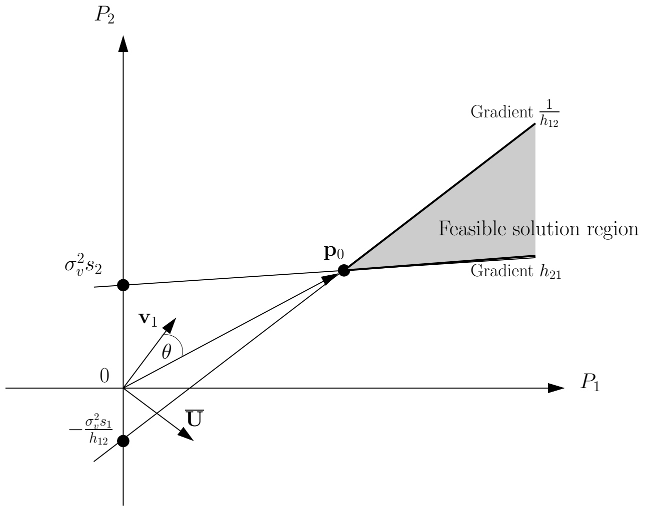

Fig. 1. An illustration of the power allocation problem in the 2-user case.

Some useful intuition can be obtained by considering the simple case of $K = 2$ users. The two user case is depicted in Figure 1 with the space of feasible solutions shaded in grey and the minimum power solution marked as $\mathbf{p}_0$. θ is the angle between the perron eigenvector $\mathbf{v}_1$ and $\mathbf{p}_0$, $\overline{\mathbf{U}}$ denotes the orthogonal complement to $\mathbf{v}_1$. Firstly, notice that we require the gradients $1/h_{12} > h_{21}$ for any feasible solution, i.e. $h_{12}h_{22} < 1$. This is simply the condition that $\lambda_{max} < 1$. By the Perron-Frobenius theory, $\mathbf{v}_1 > \mathbf{0}$ and since the solution space lies in the positive quadrant also, it is clear that a sufficiently scaled $\mathbf{v}_1$ will always be a feasible solution, although not generally optimal. In the higher dimensional case this also holds: Choose $\mathbf{p} = \alpha\mathbf{v}_1$ where α is a scalar, then (7) becomes $\alpha(1 - \lambda_{max})\mathbf{v}_1 \geq \sigma_v^2\mathbf{s}$. The element-wise inequality leads immediately to $\alpha \geq \max_k\{\frac{s_k}{v_{1k}}\}\frac{\sigma_v^2}{1-\lambda_{max}}$ for a feasible solution. We now proceed to show that $\mathbf{p}_0$ is dominated by the perron eigenvector component for large λ_{max}.

The matrix $\mathbf{H}$ can be written in terms of its Schur decomposition as $\mathbf{H} = \mathbf{U}\mathbf{T}\mathbf{U}^H$, where $\mathbf{U} = [\mathbf{v}_1, \overline{\mathbf{U}}]$ is a unitary matrix with the first column chosen to be the Perron eigenvector. $\mathbf{T}$ is an upper triangular matrix with the eigenvalues of $\mathbf{H}$ on the diagonal, $\mathbf{T} = \Lambda + \mathbf{N}$, $\Lambda = diag(\lambda_{max}, \overline{\Lambda})$, $\overline{\Lambda} = diag(\lambda_2, \ldots, \lambda_K)$ and $\mathbf{N}$ is strictly upper triangular. Since the columns of $\mathbf{U}$ span $\mathbb{R}^K$, we can express $\mathbf{s} = c_1\mathbf{v}_1 + \overline{\mathbf{U}}\overline{\mathbf{c}}$ in the basis defined by $\mathbf{U}$, where $\overline{\mathbf{c}}$ is a column vector with $K - 1$ coefficients. We can now express the optimal power allocation as

$$\begin{aligned} \mathbf{p}_0 &= \sigma_v^2(\mathbf{I} - \mathbf{H})^{-1}\mathbf{s} \\ &= \sigma_v^2\left[\frac{c_1}{1 - \lambda_{max}}\mathbf{v}_1 + \mathbf{U}(\mathbf{I} - \mathbf{T})^{-1}\mathbf{U}^H\overline{\mathbf{U}}\overline{\mathbf{c}}\right] \end{aligned} \tag{9}$$

Using the fact that $\mathbf{U}$ is unitary and writing

$$(\mathbf{I} - \mathbf{T})^{-1} = \begin{bmatrix} (1 - \lambda_{max})^{-1} & \mathbf{m}^H \\ \mathbf{0} & (\mathbf{I} - \overline{\Lambda})^{-1} + \overline{\mathbf{M}} \end{bmatrix}$$

where $(\mathbf{I} - \mathbf{T})^{-1}$ is upper triangular with the diagonal elements $(\mathbf{I} - \Lambda)^{-1}$ and the strictly upper diagonal elements in $\mathbf{m}$ and $\overline{\mathbf{M}}$, we can write $\mathbf{p}_0$ in terms of a component in the direction of $\mathbf{v}_1$ and an orthogonal component from (9) as

$$\mathbf{p}_0 = \sigma_v^2 \left[\left(\frac{c_1}{1-\lambda_{max}} + \mathbf{m}^H \overline{\mathbf{c}} \right) \mathbf{v}_1 + \overline{\mathbf{U}} \left((\mathbf{I} - \overline{\Lambda})^{-1} + \overline{\mathbf{M}} \right) \overline{\mathbf{c}} \right]$$

To establish whether the Perron eigenvector is an important component in the optimal solution, we look at the angle θ between the optimal solution and the Perron eigenvector. The smaller the angle, the closer $\mathbf{p}_0$ is to the Perron eigenvector $\mathbf{v}_1$.

$$\theta = \arccos \left(\frac{\mathbf{v}_1^H \mathbf{p}_0}{||\mathbf{v}_1||\ ||\mathbf{p}_0||} \right)$$
$$= \arccos \left(1 + \underbrace{(1-\lambda_{max})^2}_{y^2} \underbrace{\frac{||(\mathbf{I}-\overline{\Lambda})^{-1}\overline{\mathbf{c}} + \overline{\mathbf{M}}\overline{\mathbf{c}}||^2}{\left[c_1 + \mathbf{m}^H \overline{\mathbf{c}}(1-\lambda_{max})\right]^2}}_{C(\lambda_{max})} \right)^{-\frac{1}{2}}$$

where $c_1 \in \mathbb{R}$, $\mathbf{m}^H\overline{\mathbf{c}} \in \mathbb{R}$ and $||\mathbf{v}_1|| = \mathbf{v}_1^H \mathbf{v}_1 = 1$. To see how the angle behaves for large λ_{max}, we can expand θ about the point $y = 0$ while noting that $C(\lambda_{max}) \to C < \infty$ as $\lambda_{max} \to 1$ since $c_1 > 0$. Hence,

$$\theta = 0 + \sqrt{C}y - \frac{1}{3}\sqrt{C^3}y^3 + O(y^5) \approx (1-\lambda_{max})\sqrt{C} \quad (10)$$

Since $y \in [0, 1]$, the only significant contribution for large λ_{max} is the first order term and hence a large λ_{max} implies a small angle and therefore a large component of $\mathbf{p}_0$ in the direction of $\mathbf{v}_1$.

Hence, so far we have shown that an infinite number of solutions in the direction of the Perron eigenvector exist and that the optimal solution contains an eigenvector component that becomes increasingly significant with increasing λ_{max}.

Next, the load η can be written from (2), (6) and (7) in terms of $\mathbf{H}$ as

$$\eta = \frac{\mathbf{1}^T \mathbf{H} \mathbf{p}}{\mathbf{1}^T \mathbf{p}} \quad (11)$$

where $\mathbf{1}$ is a $K \times 1$ vector of all ones. In the case where $\mathbf{p} = \alpha \mathbf{v}_1$, we find $\eta = \lambda_{max}$. Therefore, for a $\mathbf{p}_0$ that is dominated by the component in the direction of $\mathbf{v}_1$, i.e. for large λ_{max}, the load η will be well approximated by λ_{max}. Conveniently, the quality of the load estimate is particularly important for high loads in which case the multi-user interference clearly dominates the noise term. Given the above observations, it is therefore reasonable to estimate the load by the Perron root, in particular for high loads. ∎

A. Estimating the load from the Perron root

Proposition 2: The Perron root λ_{max} of $\mathbf{H}$ nonnegative primitive can be estimated from a sequence that converges exponentially fast to λ_{max}. The sequence can be initialised by any $\mathbf{p} > \mathbf{0}$. The load η can be seen as a first order approximation of this sequence when initialised with the power allocation.

Proof: From (11), consider now the sequence below to estimate η from $\hat{\eta}$, for $n \geq 1$ and $\mathbf{p}(0) > \mathbf{0}$:

$$\hat{\eta}(n) = \frac{\mathbf{1}^T \mathbf{H} \mathbf{p}(n-1)}{\mathbf{1}^T \mathbf{p}(n-1)} = \frac{\mathbf{1}^T \mathbf{H}^n \mathbf{p}(0)}{\mathbf{1}^T \mathbf{H}^{n-1} \mathbf{p}(0)} \quad (12)$$

A nonnegative matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, is called *primitive* if and only if $\mathbf{A}^m > \mathbf{0}$ for some $m \geq 1$ [9]. Since all the off-diagonal elements in $\mathbf{H}$ are strictly positive, there are at most two zero terms in the summation $\left[\mathbf{H}^2\right]_{kn} = \sum_{j=1}^{K} h_{kj} h_{jn}$. Therefore, it follows that for any $K > 2$, all the elements of $\mathbf{H}^2$ are strictly positive, i.e. $\mathbf{H}^2 > \mathbf{0}$ and the matrix $\mathbf{H}$ is primitive. For any matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ that is nonnegative primitive, the following limit holds [8, Theorems 8.5.1 and 8.4.4d]:

$$\lim_{m \to \infty} \left[\lambda_{max}^{-1} \mathbf{A}\right]^m = \mathbf{L} > \mathbf{0} \quad (13)$$

where $\mathbf{L} = \mathbf{a}\mathbf{b}^T$, $\mathbf{A}\mathbf{a} = \lambda_{max}\mathbf{a}$, $\mathbf{A}^T\mathbf{b} = \lambda_{max}\mathbf{b}, \mathbf{a} > \mathbf{0}$, $\mathbf{b} > \mathbf{0}$ and $\mathbf{a}^T\mathbf{b} = 1$. The vectors $\mathbf{a}$ and $\mathbf{b}$ are the right and left eigenvector corresponding to the maximum eigenvalue λ_{max} of $\mathbf{A}$, respectively and $\mathbf{L}$ is a rank 1 matrix. Therefore, from (12) and (13), we can write

$$\lim_{n \to \infty} \hat{\eta}(n) = \lim_{n \to \infty} \left\{ \frac{\mathbf{1}^T \lambda_{max}^{-n} \mathbf{H}^n \mathbf{p}(0)}{\mathbf{1}^T \lambda_{max}^{-n} \mathbf{H}^{n-1} \mathbf{p}(0)} \right\} = \lambda_{max} \quad (14)$$

Hence, η is a first-order approximation to λ_{max} with $\mathbf{p}(0) = \mathbf{p}$. Note that the convergence of (14) holds for any initial $\mathbf{p}(0) > \mathbf{0}$. From (12), we can see also that $\mathbf{p}(n) = \mathbf{H}\mathbf{p}(n-1) = \mathbf{H}^n\mathbf{p}(0)$. Since $\lim_{n\to\infty} \hat{\eta}(n) = \lambda_{max}$, this implies that $\lim_{n\to\infty} \mathbf{p}(n)$ converges to the Perron eigenvector for any initial $\mathbf{p}(0) > \mathbf{0}$.

We remark that (14) holds whenever the limit in (13) exists, even if $\mathbf{L}$ is not of rank one, thus extending the result beyond primitive matrices. We will return to this point in Section V.

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, for any $\epsilon > 0$, there exists a constant c independent of k such that [8, 5.6.13]

$$|(\mathbf{A}^k)_{ij}| \leq c(\rho(\mathbf{A}) + \epsilon)^k \quad \forall k,\ \forall\ 1 \leq i, j \leq n \quad (15)$$

where $\rho(\mathbf{A})$ denotes the spectral radius of $\mathbf{A}$. Applying (15) to $\mathbf{1}^T\mathbf{H}^n\mathbf{p}(0)$ gives us an estimate of the rate of convergence.

$$\mathbf{1}^T \mathbf{H}^n \mathbf{p}(0) = \sum_{j=1}^{K} \sum_{i=1}^{K} (\mathbf{H}^n)_{ij} p_j(0) \leq c(\lambda_{max} + \epsilon)^n$$

Hence, the sequence in (14) converges $O(\lambda_{max}^n)$ ∎

The traditional procedure to find the load in equations (6) and (2) is to find λ_{max} and, given the feasibility condition $\lambda_{max} < 1$, solve the power allocation problem to obtain some feasible $\mathbf{p}$. Then, η can be found from (11). However, as we have seen from the proof above, we can estimate the load by estimating λ_{max}, using any initial power allocation $\mathbf{p}(0) = \mathbf{p} > \mathbf{0}$. Hence, using the proposed method, we can skip the power allocation problem altogether and reduce complexity. As noted above, the sequence of $\{\hat{\eta}\}$ converges very rapidly, thereby imposing a natural upper bound on what can be considered reasonable complexity in estimating λ_{max}.

V. Extensions to the General Non-Negative Case

In Section III, we have defined the off-diagonal elements of $\mathbf{H}$ to be strictly positive. Such an $\mathbf{H}$ corresponds to a situation where every terminal in the system interferes with every other terminal. As we have seen in Section IV above, this naturally leads to Perron-Frobenius theory since $\mathbf{H}$ is nonnegative primitive. In practise, however, it might be desirable to approximate certain off-diagonal elements in $\mathbf{H}$ with zero. In terms of the network, this would imply that certain users do not affect certain other users. For example, one could suggest that above a certain pathloss or beyond a certain separation distance, the interference between users is so small as to be negligible. Mathematically, this leads to a matrix $\mathbf{H}$ that is still nonnegative but no longer necessarily irreducible or primitive. In this Section, we therefore extend the results of Section IV to general nonnegative matrices.

A. Feasibility for general nonnegative $\mathbf{H}$

Irreducibility, as mentioned in Section III, leads to a guaranteed positive solution for $\mathbf{p} > \mathbf{0}$ by means of Perron-Frobenius theory. However, while irreducibility is a *sufficient* condition for a positive solution, it is not a necessary condition. In [4], it is shown that the feasibility condition $\rho(\mathbf{H}) = \lambda_{max} < 1$ is sufficient to ensure a positive solution for $\mathbf{p}$, given $\mathbf{s} > \mathbf{0}$ for *any* $\mathbf{H}$ nonnegative. Note that $\lambda_{max} = \rho(\mathbf{H})$ is true for any nonnegative matrix. Hence, basic feasibility is not affected by dropping the irreducibility condition.

B. Limiting behaviour of $\mathbf{H}$

To extend the convergence in (14) to the more general nonnegative case, it is necessary to consider the detailed structure of the matrix $\mathbf{H}$. The necessarily short exposition of nonnegative matrix theory that follows is primarily based on [10], [11]. In particular, we will look at three possible structures of $\mathbf{H}$; acyclic, cyclic irreducible and cyclic reducible.

In the upcoming paragraphs, general results and derivations will be demonstrated using a $n \times n$ matrix $\mathbf{A}$. These results will then be applied to $\mathbf{H}$ as it arises specifically for our application. In addition, we make the underlying assumption that we know the appropriate structure of $\mathbf{H}$. We will address the task of classifying $\mathbf{H}$ in Section V-C.

1) $\mathbf{H}$ *acyclic:* In (13), we showed convergence of the sequence $\left(\lambda_{max}^{-1}\mathbf{A}\right)^m$ for a primitive $\mathbf{A}$. Primitivity in fact ensures that $\mathbf{A}$ has only one eigenvalue of maximum modulus. We define the *period* d of $\mathbf{A}$ as the number of eigenvalues of maximum modulus so that $|\lambda_i| = \rho(\mathbf{A})$. A matrix with $d = 1$ is then called *acyclic* and if also irreducible, primitive.

For *any* acyclic matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\lim_{m\to\infty}\left[\lambda_{max}^{-1}\mathbf{A}\right]^m = \mathbf{L}$, where $\mathbf{L}$ is a rank 1 matrix. This can be seen e.g. by considering the Schur decomposition of $\mathbf{A} = \mathbf{U}\Delta\mathbf{U}^H$ where $\mathbf{U} \in \mathbb{C}^{n\times n}$ is unitary and $\Delta \in \mathbb{C}^{n\times n}$ is an upper triangular matrix with the eigenvalues on the diagonal. Assuming, without loss of generality, that eigenvalues are ordered such that $[\Delta]_{11} = \lambda_{max} = \rho(\mathbf{A})$ and given that $\left(\lambda_{max}^{-1}\mathbf{A}\right)^m = \lambda_{max}^{-m}\mathbf{U}\Delta^m\mathbf{U}^H$, it suffices to consider the convergence of $\lambda_{max}^{-m}\Delta^m = (\Xi + \Delta_0)^m$, where $\Xi = \text{diag}(\lambda_1/\lambda_{max} = 1, \lambda_2/\lambda_{max} < 1, \ldots, \lambda_n/\lambda_{max} < 1)$ and Δ_0 are the off-diagonal elements of $\lambda_{max}^{-1}\Delta$. Since $\Delta_0^m = 0 \;\; \forall\, m \geq n$ and $\lim_{m\to\infty}\Xi^m = \begin{bmatrix} 1 & \mathbf{0_{1\times(n-1)}} \\ \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times(n-1)}}\end{bmatrix}$, i.e. is an all zero matrix except for the element $(1,1)$, we see that

$$\lim_{m\to\infty}(\Xi + \Delta_0)^m = \lim_{m\to\infty}\sum_{j=1}^{n-1}\binom{m}{j}\Xi^{m-j}\Delta_0^j = \begin{bmatrix} 1 & \mathit{\Delta}_{12} & \ldots & \mathit{\Delta}_{1n} \\ 0 & \ldots\ldots\ldots & & 0 \\ \vdots & & & \vdots \end{bmatrix} = \mathit{\Delta}$$

Therefore, it is now clear that $\lim_{m\to\infty}\left[\lambda_{max}^{-1}\mathbf{A}\right]^m = \mathbf{L}$ exists for any nonnegative $\mathbf{A}$ with period $d = 1$. Hence, (14) holds for any $\mathbf{H}$ that is acyclic.

2) $\mathbf{H}$ *cyclic irreducible:* Let us now consider the case where $\mathbf{H}$ is nonnegative irreducible cyclic, i.e. with period $d > 1$. In this case, we cannot apply the method of Section V-B.1 since there are d eigenvalues of maximum modulus. Moreover, the eigenvalues of maximum modulus are given by $\lambda = \rho(\mathbf{H})e^{j\frac{2\pi}{d}k}\;\; k \in \{0, \ldots, d-1\}$.

Theorem 1: For $\mathbf{H} \in \mathbb{R}^{K\times K}$ nonnegative irreducible and cyclic (imprimitive) with period $d \geq 1$ and spectral radius $\rho(\mathbf{H}) = \lambda_{max}$, the following holds:

$$\lambda_{max}^d = \lim_{n\to\infty}\widehat{\eta^d}(n)$$

$$\widehat{\eta^d}(n) = \frac{\mathbf{1}^T\mathbf{H}^d\mathbf{p}(n-1)}{\mathbf{1}^T\mathbf{p}(n-1)} = \frac{\mathbf{1}^T(\mathbf{H}^d)^n\mathbf{p}(0)}{\mathbf{1}^T(\mathbf{H}^d)^{(n-1)}\mathbf{p}(0)}$$

for any initial $\mathbf{p}(0) \neq \mathbf{0}$.

Proof: For $\mathbf{A} \in \mathbb{C}^{n\times n}$ irreducible with period $d \geq 1$, there exists a permutation matrix $\mathbf{P}$ such that [10, p.153]:

$$\tilde{\mathbf{A}} = \mathbf{PAP}^{-1} = \begin{bmatrix} \mathbf{0} & \ldots\ldots\ldots & \mathbf{0} & \mathbf{A}_d \\ \mathbf{A}_1 & \mathbf{0} & \ldots\ldots\ldots\ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0}\ldots & \mathbf{0} \\ \vdots & & & \vdots \\ \mathbf{0} & \ldots\;\; \mathbf{0} & \mathbf{A}_{d-1} & \mathbf{0} \end{bmatrix}$$

where the diagonal blocks are square zero matrices and $\tilde{\mathbf{A}} = \mathbf{PAP}^{-1}$ is the *canonical* form of $\mathbf{A}$. Since permutations of rows and columns are a similarity transform, clearly the spectrum of $\mathbf{PAP}^{-1}$ is the same as the spectrum of $\mathbf{A}$. Further

$$\tilde{\mathbf{A}}^d = \mathbf{PA}^d\mathbf{P}^{-1} = \text{diag}(\mathbf{Q}_1, \ldots, \mathbf{Q}_d) \tag{16}$$

is a block diagonal matrix with the blocks $\mathbf{Q}_i$ nonnegative primitive, each with the same maximum eigenvalue [10]

$$\rho(\mathbf{Q}_i) = \lambda_{max}^d \;\; \forall i \in \{1, \ldots, d\} \tag{17}$$

Since all the $\mathbf{Q}_i$ blocks have the same single eigenvalue of maximum modulus, it is clear from Section V-B.1 and (17) that $\lim_{l\to\infty}[\lambda_{max}^{-d}\mathbf{Q}_i]^l = \mathbf{L}_i$ where $\mathbf{L}_i$ is a rank 1 matrix. Therefore,

$$\lim_{m\to\infty}[\lambda_{max}^{-d}\mathbf{A}^d]^m = \mathbf{P}^{-1}\lim_{m\to\infty}\left[\lambda_{max}^{-d}\text{diag}(\mathbf{Q}_1, \ldots, \mathbf{Q}_d)\right]^m\mathbf{P} = \mathbf{P}^{-1}\text{diag}(\mathbf{L}_1, \ldots, \mathbf{L}_d)\mathbf{P} = \mathbf{L}$$

where $\mathbf{L}$ is a rank d matrix. Hence, for $\mathbf{H}$ irreducible and cyclic with period d, we can write a form similar to (14), completing the proof

$$\lim_{n\to\infty}\left\{\frac{\mathbf{1}^T\lambda_{max}^{-dn}\mathbf{H}^{dn}\mathbf{p}(0)}{\mathbf{1}^T\lambda_{max}^{-dn}\mathbf{H}^{d(n-1)}\mathbf{p}(0)}\right\} = \lambda_{max}^d \tag{18}$$

■

The question of how to obtain the period d for (18), will be addressed in Section V-C.

3) $\mathbf{H}$ *cyclic reducible:* In the general nonnegative case, finally, some further steps are necessary and we can show the result given in Theorem 2:

Theorem 2: For $\mathbf{H} \in \mathbb{R}^{K\times K}$ nonnegative reducible and spectral radius $\rho(\mathbf{H}) = \lambda_{max}$, the following holds:

$$\lambda_{max}^t = \lim_{n\to\infty}\widehat{\eta^t}(n)$$

$$\widehat{\eta^t}(n) = \frac{\mathbf{1}^T\mathbf{H}^t\mathbf{p}(n-1)}{\mathbf{1}^T\mathbf{p}(n-1)} = \frac{\mathbf{1}^T(\mathbf{H}^t)^n\mathbf{p}(0)}{\mathbf{1}^T(\mathbf{H}^t)^{(n-1)}\mathbf{p}(0)}$$

for any $\mathbf{p}(0) \neq \mathbf{0}$, where t is the least common multiple of the set of periods of the diagonal blocks of the irreducible normal form of $\mathbf{H}$.

Proof: Given a matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ that is reducible we can find a permutation that results in an block upper triangular matrix with irreducible diagonal blocks $\mathbf{A}_{ii}$ with period d_i [10, p.159].

$$\mathbf{PAP}^{-1} = \begin{bmatrix} \mathbf{A}_{11} & & & \mathbf{X} \\ & \mathbf{A}_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{A}_{jj} \end{bmatrix} \quad (19)$$

We call (19) the *irreducible normal form* of $\mathbf{A}$. Since the eigenvalues of $\mathbf{A}$ are contained in the diagonal blocks $\mathbf{A}_{ii}$, i.e. $\text{spec}(\mathbf{A}) = \bigcup_{i=1}^{j} \text{spec}(\mathbf{A}_{ii})$, it follows that $\lambda_{max} = \rho(\mathbf{A}) = \max_i\{\rho(\mathbf{A}_{ii})\}$. If we denote the set of all j periods as $\mathcal{D} = \{d_1, \ldots, d_j\}$ and the least common multiple of $\mathcal{D}$ as t, it is clear that

$$\lim_{l \to \infty} [\lambda_{max}^{-t} \mathbf{A}_{ii}^{t}]^l = \begin{cases} \mathbf{L}_i & \rho(\mathbf{A}_{ii}) = \lambda_{max} \\ \mathbf{0} & \rho(\mathbf{A}_{ii}) < \lambda_{max} \end{cases} \quad (20)$$

where $\lambda_{max} = \rho(\mathbf{A})$. Therefore,

$$\lim_{n \to \infty} \left\{ \frac{\mathbf{1}^T \lambda_{max}^{-tn} \mathbf{H}^{tn} \mathbf{p}(0)}{\mathbf{1}^T \lambda_{max}^{-tn} \mathbf{H}^{t(n-1)} \mathbf{p}(0)} \right\} = \lambda_{max}^{t} \quad (21)$$

and Theorem 2 follows. ■

While somewhat more involved, obtaining the normal form of $\mathbf{H}$ is not particularly difficult and is addressed next, together with how to obtain the period.

C. Classification of $\mathbf{H}$

If presented with a given $\mathbf{H}$, it is clearly necessary to classify $\mathbf{H}$ in order to apply the techniques given above. Specifically, it is necessary to check for irreducibility or reducibility, respectively, as well as the period(s). Unless presented with a case where $\mathbf{H}$ is obviously irreducible or primitive, the best way to proceed is to construct the normal form of $\mathbf{H}$. The description that follows is based on [11, Chapter 1.2], please refer for more details. In order to make the classification easier to follow, we demonstrate the method on an example. Assume the following incidence matrix $\overline{\mathbf{H}} = [\overline{h}_{ij}]$ where a one represents an entry greater than zero in that position in $\mathbf{H}$.

$$\overline{\mathbf{H}} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (22)$$

Firstly, we need to draw the path diagram corresponding to $\overline{\mathbf{H}}$; We say that an index i leads to j if $[\overline{\mathbf{H}}^k]_{ij} = \overline{h}_{ij}^{(k)} > 0$ for some integer $k \geq 1$. If an index i leads to j and j also leads to i, the indices i, j are said to communicate. Start with $i = 1$ (we call this stage 0) and draw a connection to the connected indices (2 and 4 in Figure 2a, we call this stage 1). From each of these indices, draw the connections to their connected indices etc. When an index is encountered that already occurred at an earlier stage, stop the branch at this index. In Figure 2a, the resulting path diagram is shown. Notice that only indices

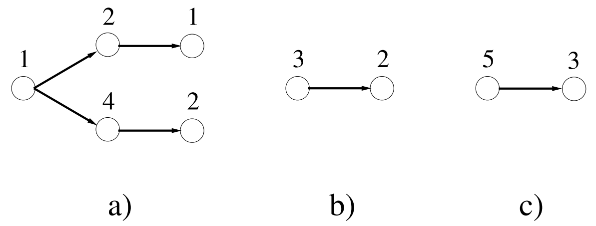

Fig. 2. The path diagrams for example matrix $\mathbf{H}$.

$\{1, 2, 4\}$ have occurred so far. If all the indices occur in the first path diagram, then every index leads to every other index. In this case, the matrix $\overline{\mathbf{H}}$ (and therefore $\mathbf{H}$) is irreducible because the graph is strongly connected and all that is left to determine is the period of $\mathbf{H}$. Returning to the example of $\mathbf{H}$, it is clear that $\mathbf{H}$ is reducible since only indices $\{1, 2, 4\}$ have occured in Figure 2a. Start a new path diagram with the lowest index not yet encountered and proceed as described above. In addition, a branch also stops if an index occurs that is already part of an earlier path diagram. This leads to Figures 2b and 2c. At this point we need to define *essential* and *inessential* classes of indices for each path diagram. An essential class is a set of indices where every index in the class communicates but cannot lead to an index outside the class. Otherwise, it is an inessential class of indices. Using this classification, we see from Figure 2a-c that $\{1, 2, 4\}$ is an essential class while $\{3\}$ and $\{5\}$ are inessential classes. We can now construct the canonical form through the simultaneous permutation of rows and columns according to the index class. Reorder the indices from inessential to essential and within each class from high to low so that the new index ordering is changing from $\{1, 2, 3, 4, 5\}$ to $\{5, 3, 4, 2, 1\}$. Therefore

$$\mathbf{P}\overline{\mathbf{H}}\mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

which can now be seen to be block upper triangular with irreducible diagonal blocks as in (19) with $\mathbf{A}_{11} = \mathbf{A}_{22} = [0]$ and

$$\mathbf{A}_{33} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

It remains to determine the period of an irreducible matrix. Fortunately, this is relatively straightforward. The algorithm suggested here is from [10]. For an irreducible matrix $\mathbf{A} = [a_{ij}] \in \mathbb{C}^{n \times n}$, the *patterns* of elements $a_{ij} > 0$ and elements $a_{ij} = 0$ repeat with period d between powers of $\mathbf{A}$. Denoting the j-th column of matrix $\mathbf{A}^l$ by $\mathbf{a}_j^{(l)}$ and noting that $\mathbf{a}_j^{(l+1)} = \mathbf{A}\mathbf{a}_j^{(l)}$, it suffices to look at how the pattern of zero elements (or non-zero elements) evolves of any single column, whereby the choice of column is irrelevant. Note that since only the pattern but not the actual values are of importance, this algorithm can be applied using boolean operations on $\overline{\mathbf{H}}$. For any $\mathbf{a}_j^{(m)}$, hence, compare the zero-pattern with the zero-pattern of all $\mathbf{a}_j^{(k)}$ where $1 \leq k < m$. When the same zero pattern is encountered, say for $\mathbf{a}_j^{(m_0)}$ and $\mathbf{a}_j^{(k_0)}$, then the period is $d = k_0 - m_0$. For the example given above, the period is $d(\mathbf{A}_{11}) = d(\mathbf{A}_{22}) = d(\mathbf{A}_{33}) = 1$.

Hence, we have established how to classify the matrix $\mathbf{H}$ and convert it to irreducible normal form, if required, as well as how to easily determine the period of an irreducible matrix.

VI. Other Methods to Easily Estimate the Perron Root

In this Section, we take a brief look at various other methods to estimate the Perron root, λ_{max}, in order to estimate the load. The focus is on methods that are sufficiently simple to allow analytical treatment or to be of practical use and that rely essentially only on the matrix $\mathbf{H}$ itself and the information contained therein.

A. Gershgorin theorem

The first theorem useful to estimate λ_{max} is the well-known Gershgorin Theorem [8], [9]. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, then

$$\text{spec}(\mathbf{A}) \subset \bigcup_{i=1}^{n} \left\{ \lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}| \triangleq r_i \right\}$$

where spec($\mathbf{A}$) denotes the spectrum of $\mathbf{A}$, λ denotes an eigenvalue of $\mathbf{A}$, and a_{ij} is element (i, j) of $\mathbf{A}$.

Since $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}^T$, the application of the Gershgorin theorem to $\mathbf{A}^T$ shows that the theorem also holds for column sums, which we shall denote by $c_i = \sum_{j=1,j\neq i}^{n} |a_{ji}|$.

In our case, the diagonal elements are all zero, therefore all the Gershgorin circles are centred at the origin. Hence

$$\lambda_{max}(\mathbf{H}) \leq \min\left\{ \max_{i\in\{1,\ldots,K\}} \{r_i\}, \max_{l\in\{1,\ldots,K\}} \{c_l\} \right\} \quad (23)$$

In the downlink single cell scenario, it can be shown that the estimate resulting from the column sums is always tighter than the estimate from the row sums [12].

A very interesting observation can be made for the case of $\mathbf{H}$ nonnegative reducible. Recalling from Section V-B.3 that the eigenvalues of $\mathbf{H}$ are contained in the diagonal blocks $\mathbf{H}_{ii}$ of the irreducible normal form, it is clear that we can apply the Gershgorin theorem to the diagonal blocks of the normal form only, i.e.

$$\lambda_{max}(\mathbf{H}) \leq \min\left\{ \max_{n} \left[\max_{i} \{r_i(\mathbf{H}_{nn})\} \right], \max_{n} \left[\max_{i} \{c_i(\mathbf{H}_{nn})\} \right] \right\} \quad (24)$$

Clearly, this is always at least as tight a bound as the Gershgorin theorem applied directly to $\mathbf{H}$. Therefore, whenever $\mathbf{H}$ is reducible, (24) should be applied.

B. Brauer's Cassini Ovals

Another bound on the maximum eigenvalue can be obtained from *Brauer's Cassini Ovals* inclusion regions for eigenvalues [8], [9]. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, then

$$\text{spec}(\mathbf{A}) \subset \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{\lambda \in \mathbb{C} : |\lambda - a_{ii}||\lambda - a_{jj}| \leq r_i r_j\}$$

Again, as for the Gershgorin discs, since spec($\mathbf{A}$) = spec($\mathbf{A}^T$), there exists a column-sum version of the Brauer's Cassini ovals and therefore

$$\lambda_{max}^2 \leq \min\left\{ \max_{i,j;i\neq j} \{r_i r_j\}, \max_{i,j;i\neq j} \{c_i c_j\} \right\} \quad (25)$$

In our case, since e.g. $r_i \geq r_j$, it is immediately clear that the bound provided by the Brauer's Cassini ovals must be *at least as good* as the Gershgorin bound. Indeed, this result is true in general, see e.g. [13].

As for the Gershgorin theorem, Brauer's bound can potentially improve when applied to the diagonal blocks $\mathbf{H}_{ii}$ of the irreducible normal form of $\mathbf{H}$. Therefore, for a reducible $\mathbf{H}$, the following should be applied:

$$\lambda_{max}^2 \leq \min\left\{ \max_{n} \left[\max_{i,j;i\neq j} \{r_i(\mathbf{H}_{nn}) r_j(\mathbf{H}_{nn})\} \right], \max_{n} \left[\max_{i,j;i\neq j} \{c_i(\mathbf{H}_{nn}) c_j(\mathbf{H}_{nn})\} \right] \right\}$$

Again, this will be at least as good as the Brauer Cassini Ovals applied to $\mathbf{H}$ directly.

C. Brualdi lemniscate sets

From the Gershgorin discs and Brauer's Cassini Ovals the question arises whether the product of more than two row/column-sums would further improve the eigenvalue estimate. Unfortunately, products of three or more row/column-sums are not generally eigenvalue inclusion regions [14]. However, for weakly irreducible (and hence also irreducible) matrices, there exists a result called the *Brualdi lemniscate set* [8], [14]. The Brualdi lemniscate set requires one to find all the non-trivial cycles in $\mathbf{H}$ and allows to use row/colum products with as many factors as the cardinality of the non-trivial path. Generally speaking, the application of Brualdi's lemniscate set is, however, more tedious and will in many instances provide the same bound as the Cassini Ovals. In particular, in the case of $\mathbf{H}$ without off-diagonal zero elements, the Brualdi lemniscate set collapses back into Brauer's Cassini Ovals [12].

Based on this observation and the complexity associated with finding and evaluating the cycles, this method is not further pursued here.

VII. Numerical Evaluation

In this Section, the previously proposed methods to estimate the Perron root are compared numerically with the system load in a downlink scenario.

A. System Model

The cellular system is modelled as a 2 km by 2 km square area. Assuming a simple pathloss model given by $L(d) = Wd^{-n}$ where d is the distance and $W = \left(\frac{\lambda}{4\pi}\right)^2$ a frequency dependent constant and $n = 3$ is the assumed pathloss exponent. λ is the wavelength corresponding to an assumed transmission frequency of 2 GHz. Each base station is assumed to cover a circular area corresponding to a theoretical maximum pathloss of 120 dB at the cell edge. In this particular example, this leads to $M = 5$ base stations necessary to theoretically cover the area. These M base stations as well as K users are placed randomly in the area, uniformly distributed. Each user is associated with the closest base station, corresponding to the smallest pathloss between any of the base stations and the user. The matrix $\mathbf{H}$ and the vector $\mathbf{s}$ are then constructed according to (8). To model different user requirements, each of the K users is assigned either a voice channel target SINR (12.2 kbps) with a probability of 80% or a data channel target SINR (384 kbps). Assumed are $S_{voice} = -16$ dB and $S_{data} = -5.5$ dB at chip level, respectively. The orthogonality factor α_k is chosen from a uniform distribution between 0.2 and 0.8, modelling different multipath environments. For every K users, 500 realisations are computed.

In what follows, the matrix $\mathbf{H}$ is nonnegative primitive, i.e. with no off-diagonal zeros. As we have seen this case arises naturally and the treatment is nearly identical with the irreducible case with period $d > 1$.

The algorithms of Section VI for reducible matrices are highly dependent on the actual structure of $\mathbf{H}$. The performance gain of the Gershgorin and Brauer theorems for reducible matrices over the standard version of the algorithms depends strongly on how 'disconnected' $\mathbf{H}$ is, i.e. on the number of diagonal blocks in the irreducible normal form. Since the primitive case presented here provides a lower bound to the performance obtainable for a general non-negative matrix, the reducible case is only briefly treated towards the end of this section.

Clearly, as the number of users K increases, so does the load. To ensure a theoretically feasible $\mathbf{H}$ with $\lambda_{max} < 1$, users are dropped, if necessary, until the feasibility condition is met. Based on the observations made in Sections IV and VI, the users causing the largest row sums in $\mathbf{H}$ are dropped. Note that this is by no means claimed to be optimal access control.

B. Numerical results

In Figure 3, the Perron root and the different load estimates are compared with the load of a nonnegative primitive $\mathbf{H}$. The downlink load according to (11) is computed with the optimal power allocation given by $\mathbf{p}_0 = (\mathbf{I}_K - \mathbf{H})^{-1}\sigma_v^2\mathbf{s}$. The iterative method given in (12), (14) is shown for $n = \{1, 2, 5\}$ and initialised with $\mathbf{p}(0) = \mathbf{1}$. Also shown are the Gershgorin estimate and the Brauer estimate from Sections VI-A and VI-B, respectively. It can be seen that the Gershgorin and Brauer estimates are consistently poor, over the entire range of load values. The Brauer estimate is essentially identical to the Gershgorin estimate except for very small K, as expected. The iterative sequence estimate proposed in Section IV-A, however, is quite close to the true load value in absolute terms, even for small loads and even for $n = 1$. More importantly, Figure 3 shows that the Perron root is a good approximation of the load. Bearing in mind the context of early network planning and the fact that all assumed parameters in the model are approximations in practise, the estimate of the load obtained with the iterative sequence estimator is better than $\eta \approx \hat{\eta} \pm 0.08$, which is quite accurate for network planning purposes.

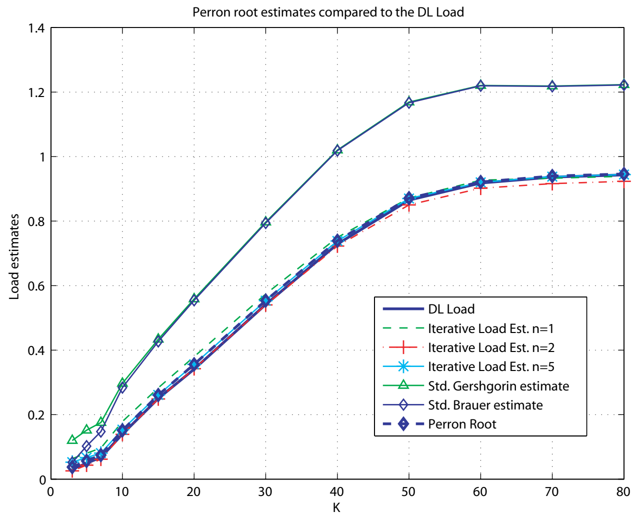

Fig. 3. This figure compares the load, the Perron root and the different load estimation techniques.

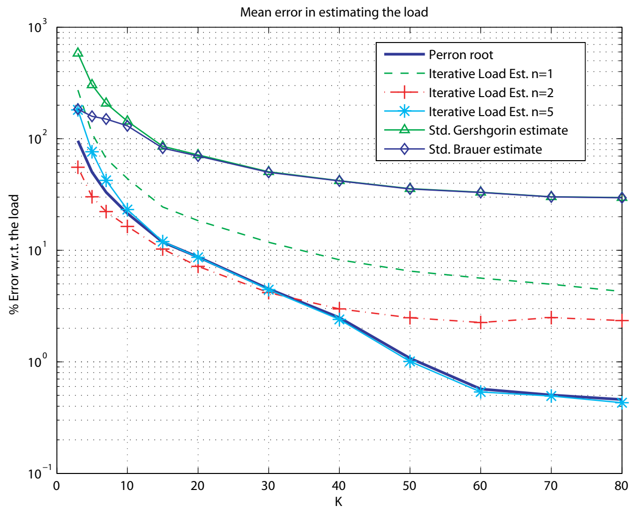

Fig. 4. This figure shows the error of the various load estimation techniques, normalised with respect to the load.

In Figure 4, the error of the load estimates is normalised with respect to the load, allowing a more detailed error analysis. As expected, the Gershgorin and Brauer estimates of the load are not very useful and never get below about a 30% error. While all the techniques are considerably off for very small loads ($K < 10$) due to the nearly constant absolute error that can be observed in Figure 3, the estimates get rapidly better for larger load values. The error between the Perron root and the load, as well as the load and the iterative sequence estimates for $n = \{2, 5\}$ are below 10% from about 18 users onwards which corresponds to a load of approximately $30 \pm 3\%$. For $n = 1$, the 10% barrier is only broken at 35 users, corresponding to a load of about $65 \pm 7\%$. Beyond $K = 40$, the error of the Perron root and the estimates for $n = \{2, 5\}$ are at or below 3% for loads of $\eta \geq 75\%$. At this point, even for $n = 1$, the error is less than 8%.

Hence, as expected, the accuracy of the estimates increases rapidly with increasing load. Regarding the complexity, $n = 2$ seems to be the best complexity-accuracy trade-off. If one can live with larger errors, even $n = 1$ might be worth considering.

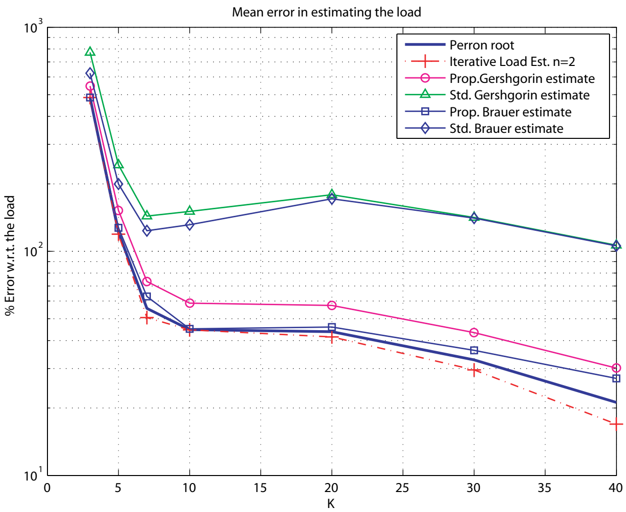

Fig. 5. This figure shows the error of the load estimation techniques for reducible $\mathbf{H}$, normalised with respect to the load.

On the other hand, for early network planning purposes, $n = 5$ seems too costly in light of the moderate improvements over $n = 2$.

For the reducible case, elements in the irreducible Matrix $\mathbf{H}$ are removed until it is sparse enough to become reducible. As mentioned above, the irreducible case is a lower bound to the performance for the estimation techniques, so the main interest is to verify the performance improvements obtainable for the Gershgorin and Brauer estimates for reducible $\mathbf{H}$. In Figure 5, the standard forms of the Brauer and Cassini methods are compared with the proposed method of applying the Brauer and Cassini estimates on the diagonal blocks of the irreducible normal form. As can be seen, the estimates are significantly improved. Note however that the load for e.g. $K = 40$ is only about 35% because a lot of elements had to be removed. For a general matrix $\mathbf{H}$, however, there is no guarantee that the matrix will be reducible and even if, into how many diagonal block matrices. Hence the Gershgorin and Brauer methods are very inconsistent because they depend so strongly on the structure of $\mathbf{H}$. In addition, the iterative method still performs better for all n. Only $n = 2$ is shown in the Figure to reduce clutter.

In summary, the iterative sequence method of Sections IV and V is clearly the most suitable approach for practical applications. For a rough approximation of the load, $n = 1$ is already sufficient whereas $n = 2$ provides an already very good approximation at reasonable complexity, without the need to solve the optimal power allocation problem. The Gershgorin and Brauer estimates, on the other hand, provide bounds that are too loose in general. Compared to the iterative solution, however, these are very general eigenvalue bounds that do not impose any particular structure or properties on $\mathbf{H}$. Therefore, it was to be expected that they perform less well for the scenario considered.

VIII. Conclusions

We have shown how the concept of load as used for early-stage CDMA network planning, is related to the Perron root arising in the power allocation problem and propose to use the Perron root to estimate the load. Several methods to estimate the Perron root have been introduced and compared. As opposed to the classical load formula, these methods do not require the solution to the power allocation problem to compute the load. In particular, we found that the proposed iterative approximation to the Perron root provides a significantly better estimate than those obtained using Gershgorin's theorem and Brauer's Cassini Ovals at low complexity.

Acknowledgement

The author would like to thank Dr Daniela Tuninetti for useful discussions, notably on nonnegative matrix theory.

References

[1] S. V. Hanly, "Congestion Measures in DS-CDMA Networks," *IEEE Trans. Commun.*, vol. 47, Mar. 1999.
[2] J. Laiho, A. Wacker, and T. N. (Ed), *Radio Network Planning and Optimisation for UMTS*. Wiley, 2002.
[3] S. Stanczak and H. Boche, "Information Theoretic Approach to the Perron Root of Nonnegative Irreducible Matrices," in *Proc. IEEE Information Theory Workshop (ITW 2004)*, San Antonio, USA, Oct. 2004.
[4] L. I. R. Mathar, "The Geometry of the Capacity Region for CDMA Systems with General Power Constraints," *IEEE Trans. Wireless Commun.*, 2006, to appear.
[5] H. Holma and A. Toskala, *WCDMA for UMTS: Raio Access for Third Generation Mobile Communications*. Wiley, 2002.
[6] E. G. Lundin, "Uplink load in cdma cellular systems," Ph.D. dissertation, Linkoping University, Sweden, Oct. 2003.
[7] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge University Press, 2005.
[8] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.
[9] D. S. Bernstein, *Matrix Mathematics*. Princeton University Press, 2005.
[10] A. Graham, *Nonnegative Matrices and applicable Topics in Linear Algebra*. Chichester, UK: Ellis Horwod Ltd., 1987.
[11] E. Seneta, *Non-negative Matrices and Markov Chains*. Springer-Verlag, 1973.
[12] C. Fischer, "On CDMA Load and the Perron Root," in *Proc. 19th International Zurich Seminar on Communications*, Zurich, Switzerland, Feb. 2006.
[13] R. S. Varga and A. Krautstengl, "On Gersgorin-type problems and Ovals of Cassini," *Electron. Trans. Numerical Analysis*, vol. 8, pp. 15–20, 1999.
[14] R. S. Varga, "Gersgorin-type eigenvalue inclusion theorems and their sharpness," *Electron. Trans. Numerical Analysis*, vol. 12, pp. 113–133, 2001.

Christian Fischer received a Master of Engineering degree in Electronic Engineering from the University of Southampton (UK) in 1998 and a Doctorate from Ecole Nationale Superieure des Telecommunications (ENST Paris)/Institut Eurecom (France) in 2002 for his research on signal processing for CDMA/UMTS. He subsequently joined Swisscom Innovations (Switzerland), where he was a Senior Research Engineer for emerging wireless systems and lead Ultra-Wideband activities. Since 2005, he is heading the telecommunications engineering activities of Fischer AG, Windisch, Switzerland and is an Adjunct Assistant Professor at the University of Illinois at Chicago, USA. His main research interests are communications and signal processing applied to emerging wireless technologies, Ultra-Wideband Systems, Multiple Antenna Systems, WiMAX and Cellular Systems.